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On Abelian groups with a certain property on essential subgroups

by

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In the lecture note [1] (cf. the references at the end of this paper) by C. Faith, the condition that the singular submodule $Z(M)$ of a module M is zero plays a very important role. Such a module satisfies a certain property on essential subgroups, as will be stated in Proposition 1. The main object of this paper is to characterize Abelian groups with this property.

PROPOSITION 1. *Let M be a right R-module (R a ring) with the singular submodule $Z(M)=0$. If A and B are essential submodules of submodules A' and B' of M, respectively, then $A+B$ is an essential submodule of $A'+B'$.*

Proof. Assume that $c=a'+b'$ ($a'\in A'$, $b'\in B'$) is a nonzero element of $A'+B'$. Consider the submodules

$$a'_A=\{r\in R\,|\,a'r\in A\} \quad \text{and} \quad b'_B=\{r\in R\,|\,b'r\in B\}$$

of the right module $R=R_R$. Since A is essential in A', a'_A is essential in R. Similarly, b'_B is essential in R. Thus, $I=a'_A\cap b'_B$ is essential in R. Using $Z(M)=0$, we have $cI\neq 0$, and therefore $\{c\}\cap(A+B)\neq 0$, where $\{c\}$ denotes the submodule generated by c. Hence, $A+B$ is essential in $A'+B'$.

Remark 1. It follows easily from this proposition that, for any module M with $Z(M)=0$, each submodule N of M is contained in a unique closed essential extension N' contained in M (cf. p. 61 of Faith's book [1]).

From now on, we shall consider only Abelian groups. Thus, a group is an Abelian group throughout this paper.

In this paper, for brevity, a group G is called an E-group when, if A and B are essential subgroups of subgroups A' and B' of G respectively, then $A+B$ is an essential subgroup of $A'+B'$. By Proposition 1, if $Z(G)=0$, then G is an E-group. Therefore, any torsion-free group is an E-group. (Concering the terminologies on Abelian groups, cf. Fuchs [2].)

Remark 2. It does not hold in general that $Z(G)=0$ follows from the assumption that G is an E-group. In fact, we can easily prove that, for any torsion group G, it holds $Z(G)=G$, while there exists a nontrivial torsion E-group (for instance, quasicyclic group).

Our main result is the following one.

THEOREM. *A group G is an E-group if and only if G is either a torsion-free group or a torsion group whose every p-primary component G_p has one of the following three types of groups:*
1. *G_p is elementary,*
2. *G_p is cyclic,*
3. *G_p is quasicyclic.*

In order to prove this theorem, we shall prove the following propositions.

PROPOSITION 2. *Any subgroup of an E-group is also an E-group.*

PROPOSITION 3. *A torsion group G is an E-group if and only if every p-primary component G_p of G is an E-group.*

These two propositions are obvious by the definition of E-groups.

PROPOSITION 4. *Any mixed group G is not an E-group.*

Proof. Let T be the torsion part of G, and take two nonzero elements u, v in G such that $u \notin T$ and $v \notin T$. (Since G is mixed, there are such elements.) Let n be the order of v. Consider the cyclic subgroups $A=\{nu\}$, $A'=\{u+v\}$, $B=\{u\}$, $B'=\{u\}$. Then, A and B are essential in A' and B', respectively. However, $A+B=B=\{u\}$ is not essential in $A'+B'=\{u\}\oplus\{v\}$. Thus, G is not an E-group.

Since any torsion-free group is an E-group, the rest of the proof of our theorem is to prove that a p-group G is an E-group if and only if G is one of the three types of groups stated in the theorem. The 'if' part is quite obvious. On the other hand, a p-group G, which is not any of the three types of groups, contains a subgroup $H=\{u\}\oplus\{v\}$, where u and v are elements of order p^2 ane p, respectively. Hence, it remains for us to prove the following final proposition.

PROPOSITION 5. *Assume that $G=\{u\}\oplus\{v\}$, where u and v are elements of order p^2 and p, respectively. Then, G is not an E-group.*

Proof. Consider the cyclic subgroups $A=\{pu\}$, $A'=\{u\}$, $B=\{pu\}$, $B'=\{u+v\}$. Then A and B are essential in A' and B', respectively. However $A+B=A=\{pu\}$ is not essential in $A'+B'=\{u\}\oplus\{v\}=G$. Thus, G is not an E-group.

Therefore, the proof of our theorem has come to an end.

References

[1] FAITH, C.; *Lectures on Injective Modules and Quotient Rings*, Springer, 1967.
[2] FUCHS, L.; *Infinite Abelian Groups*, 1, Academic Press, 1970.

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